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LETTER TO THE EDITOR

Infinite statistics and the $SU(1, 1)$ phase operator

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Abstract

A few years ago, Agarwal (1991 *Phys. Rev.* A **44** 8398) showed that the Susskind–Glogower phase operators, expressible in terms of Bose operators, provide a realization of the algebra for particles obeying infinite statistics. In this paper we show that the $SU(1, 1)$ phase operators, constructed in terms of the elements of the $su(1, 1)$ Lie algebra, also provide a realization of the algebra for infinite statistics. There are many realizations of the $su(1, 1)$ algebra in terms of single or multimode bose operators, three of which are discussed along with their corresponding phase states. The Susskind–Glogower phase operator is a special case of the $SU(1, 1)$ phase operator associated with the Holstein–Primakoff realization of $su(1, 1)$.

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A few years ago, Agarwal [1] presented, in terms of a set of boson operators, a realization of the annihilation and creation operators for particles satisfying infinite statistics whereby any representation of the symmetric group can occur. The algebra of such operators was discussed by Greenberg [2] following a suggestion of Hegstrom. If $\hat{b}$ and $\hat{b}^\dagger$ represent the annihilation and creation operators of such particles, they must satisfy the relations

$$\hat{b}\hat{b}^\dagger = \hat{1}, \qquad \hat{b}^\dagger\hat{b} \neq \hat{1}, \tag{1}$$

where $\hat{1}$ represents the identity operator. Agarwal [1] pointed out that these relations are satisfied by the operators

$$\hat{b} = (\hat{1} + \hat{a}^\dagger\hat{a})^{-1/2}\hat{a}, \qquad \hat{b}^\dagger = \hat{a}^\dagger(\hat{1} + \hat{a}^\dagger\hat{a})^{-1/2}, \tag{2}$$

where the operators $\hat{a}$ and $\hat{a}^\dagger$ are bose operators satisfying the usual commutation relations $[\hat{a}, \hat{a}^\dagger] = \hat{1}$. It is easy to show, using equation (2), that

$$\hat{b}|n\rangle = |n-1\rangle(1-\delta_{n0}), \qquad \hat{b}^\dagger|n\rangle = |n+1\rangle, \qquad \hat{b}^\dagger\hat{b}|n\rangle = |n\rangle(1-\delta_{n0}), \tag{3}$$

where $|n\rangle$ is an eigenstate of the bose number operator $\hat{a}^\dagger\hat{a}$. An alternative way of representing the operators $\hat{b}$ and $\hat{b}^\dagger$ is through the decompositions

$$\hat{b} = \sum_{n=1}^{\infty}|n-1\rangle\langle n|, \qquad \hat{b}^\dagger = \sum_{n=1}^{\infty}|n\rangle\langle n-1| \tag{4}$$

from which it is apparent that

$$\hat{b}\hat{b}^{\dagger} = \hat{1}, \qquad \hat{b}^{\dagger}\hat{b} = \hat{1} - |0\rangle\langle 0|. \tag{5}$$

The number operator constructed by Greenberg satisfies

$$\hat{N} = \sum_{m=1}^{\infty} \hat{b}^{\dagger m}\hat{b}^{m} = \hat{a}^{\dagger}\hat{a}, \qquad \hat{N}|n\rangle = n|n\rangle, \tag{6}$$

as can easily be verified using equation (4).

The operators of equation (1) are 'one-way' unitary as are the 'exponential' phase operators discussed by Susskind and Glogower (SG) [3] to describe the phase properties of a single-mode quantized optical field. The SG phase operators, given as

$$\hat{e}_{\mathrm{SG}} = (\hat{a}\hat{a}^{\dagger})^{-1/2}\hat{a}, \qquad \hat{e}_{\mathrm{SG}}^{\dagger} = \hat{a}^{\dagger}(\hat{a}\hat{a}^{\dagger})^{-1/2}, \tag{7}$$

are identical to the operators $\hat{b}$ and $\hat{b}^{\dagger}$, respectively, as can be seen upon using the commutation relations for bose operators: $\hat{a}\hat{a}^{\dagger} = \hat{1} + \hat{a}^{\dagger}\hat{a}$. Furthermore, there exist coherent states of the operator $\hat{e}_{\mathrm{SG}}$, the phase coherent states $|z\rangle$ such that $\hat{e}_{\mathrm{SG}}|z\rangle = z|z\rangle$ where

$$|z\rangle = (1 - |z|^2)^{1/2} \sum_{n=0}^{\infty} z^n |n\rangle, \qquad z \in \mathbb{C}, \qquad 0 \leqslant |z| < 1. \tag{8}$$

These states have been known since at least 1973 [4] though they have occasionally been rediscovered [5].

In 1988, the present author [6] constructed phase operators associated with the Lie algebra $su(1,1)$. It is well known that the $su(1,1)$ Lie algebra and the associated Lie group $SU(1,1)$ play a role in a number of important quantum mechanical systems and, in particular, in quantum optics. The Lie algebra consists of the operators $\{\hat{K}_0, \hat{K}_{\pm}\}$ satisfying the commutation relations

$$[\hat{K}_0, \hat{K}_{\pm}] = \pm\hat{K}_{\pm}, \qquad [\hat{K}_+, \hat{K}_-] = -2\hat{K}_0, \tag{9}$$

and the Casimir operator $\hat{C}$ given by

$$\hat{C} = \hat{K}_0^2 - \tfrac{1}{2}(\hat{K}_+\hat{K}_- + \hat{K}_-\hat{K}_+). \tag{10}$$

The relevant unitary irreducible representations are the positive discrete series $\mathcal{D}_k^+$: $\{|k,m\rangle, m = 0, 1, 2, \ldots\}$ where

$$\hat{K}_0|k,m\rangle = (m+k)|k,m\rangle, \tag{11}$$

$$\hat{C}|k,m\rangle = k(k-1)|k,m\rangle, \tag{12}$$

$$\hat{K}_+|k,m\rangle = [(m+1)(m+2k)]^{1/2}|k,m+1\rangle, \tag{13}$$

$$\hat{K}_-|k,m\rangle = [m(m+2k-1)]^{1/2}|k,m-1\rangle. \tag{14}$$

The number k is the Bargmann index which, for the standard unitary irreducible representations, takes the values $k = \frac{1}{2}, 1, \frac{3}{2}, \ldots$. Some realizations of the $su(1,1)$ algebra require certain non-standard representations. The single-mode 'two-boson' realization [7] requires the representations for which $k = \frac{1}{4}$ and $\frac{3}{4}$, which are actually representations of the covering group $\widetilde{SU}(1,1)$. The states $|k,m\rangle$, of course, form a complete set according to

$$\sum_{m=0}^{\infty} |k,m\rangle\langle k,m| = \hat{1}_k, \tag{15}$$

where $\hat{1}_k$ is the identity operator within the space $\mathcal{D}_k^+$.

The $SU(1,1)$ phase operators are constructed as follows:

$$\hat{e}_k = (\hat{K}_-\hat{K}_+)^{-1/2}\hat{K}_-, \qquad \hat{e}_k^\dagger = \hat{K}_+(\hat{K}_-\hat{K}_+)^{-1/2}, \tag{16}$$

which satisfy

$$[\hat{K}_0, \hat{e}_k] = -\hat{e}_k, \qquad \left[\hat{K}_0, \hat{e}_k^\dagger\right] = \hat{e}_k^\dagger. \tag{17}$$

Noting that

$$\hat{e}_k|k,m\rangle = |k,m-1\rangle(1-\delta_{m0}), \qquad \hat{e}_k^\dagger|k,m\rangle = |k,m+1\rangle, \tag{18}$$

we may write

$$\hat{e}_k = \sum_{m=1}^{\infty}|k,m-1\rangle\langle k,m|, \qquad \hat{e}_k^\dagger = \sum_{m=1}^{\infty}|k,m+1\rangle\langle k,m|. \tag{19}$$

It is evident that $e_k e_k^\dagger = \hat{1}_k, e_k^\dagger e_k = \hat{1}_k - |k,0\rangle\langle k,0|$ and thus we have again operators describing particles with infinite statistics. The corresponding Greenberg number operator is

$$\hat{N}_k = \sum_{n=1}^{\infty}\hat{e}_k^{\dagger n}\hat{e}_k^n = \hat{K}_0 - k\hat{1}_k, \qquad \hat{N}_k|k,m\rangle = m|k,m\rangle. \tag{20}$$

The $SU(1,1)$ phase states $|\eta,k\rangle$ are given as eigenstates of the phase operator $\hat{e}_k$ according to $\hat{e}_k|\eta,k\rangle = \eta|\eta,k\rangle$ where

$$|\eta,k\rangle = (1-|\eta|^2)^{1/2}\sum_{m=0}^{\infty}\eta^m|k,m\rangle, \qquad \eta\in\mathbb{C}, \qquad 0\leqslant|\eta|<1. \tag{21}$$

Note the expansion coefficients are independent of the Bargmann index. Vourdas [8] has studied the properties of these phase states.

We now examine some specific realizations of the $su(1,1)$ algebra and corresponding representations. We first consider the Holstein–Primakoff [9] realization of $su(1,1)$ given by [4, 10]

$$\hat{K}_+(k) = \hat{a}^\dagger(2k+\hat{a}^\dagger\hat{a})^{1/2}, \qquad \hat{K}_-(k) = (2k+\hat{a}^\dagger\hat{a})^{1/2}\hat{a}, \qquad \hat{K}_0(k) = \hat{a}^\dagger\hat{a}+k \tag{22}$$

where the operators $\hat{a}$ and $\hat{a}^\dagger$ are again bose operators and where it is to be understood that the terms with index k are really $k\hat{1}$. For *every* allowed value of k, the basis of the corresponding UIR is the same: the number states $|n\rangle_a$ associated with bose operators $\hat{a}$ and $\hat{a}^\dagger$. That is, for the HP realization the relevant representations are $\mathcal{D}_k^+ : \{|k,m\rangle = |m\rangle_a; m = 0,1,2\ldots\}$. The corresponding phase coherent states are independent of k. For the special case where $k = 1/2$ (and only for this case), the $su(1,1)$ phase operators go over to the SG phase operators:

$$\hat{e}_{\mathrm{SG}} = \hat{K}_-(1/2), \qquad \hat{e}_{\mathrm{SG}}^\dagger = \hat{K}_+(1/2). \tag{23}$$

The corresponding phase coherent states, which, of course, are identical to those of equation (8) if $z = \eta$, are just the $SU(1,1)$ Barut–Girardello coherent states [11], i.e. eigenstates of the operator $\hat{K}_-$, for the Holstein–Primakoff realization (for $k = 1/2$).

Another important realization of $su(1,1)$ is the two-boson (or non-degenerate 'two-photon') realization [7, 12]

$$\hat{K}_+ = \hat{c}^\dagger\hat{d}^\dagger, \qquad \hat{K}_- = \hat{c}\hat{d}, \qquad \hat{K}_0 = \tfrac{1}{2}(\hat{c}^\dagger\hat{c}+\hat{d}^\dagger\hat{d}+1), \tag{24}$$

where the $\hat{c}$ and $\hat{d}$ are boson operators and the Casimir operator for this realization is

$$\hat{C} = \tfrac{1}{4}(\hat{\Delta}^2-1), \qquad \hat{\Delta} = \hat{c}^\dagger\hat{c}-\hat{d}^\dagger\hat{d}. \tag{25}$$

If q, the degeneracy parameter, represents the eigenvalue of the operator $\hat{\Delta}$, then the Bargmann index takes the values $k = (1+|q|)/2, |q| = 0,1,2\ldots$, and thus we have the standard representations. Without loss of generality, we may assume that $q \geqslant 0$ meaning that the number of particles of type c is q greater than of type d. In this case the relevant UIRs of $su(1,1)$ are $\mathcal{D}_k^+ : \{|k,m\rangle = |n+q\rangle_c \otimes |n\rangle_d\}$ where $k = (1+q)/2$ and $m = n+k$, as can easily be checked. The phase operators for this realization are

$$\begin{aligned} \hat{e}_k &= (\hat{c}\hat{d}\hat{c}^\dagger\hat{d}^\dagger)^{-1/2}\hat{c}\hat{d} = [(1+\hat{c}^\dagger\hat{c})(1+\hat{d}^\dagger\hat{d})]^{-1/2}\hat{c}\hat{d}, \\ \hat{e}_k^\dagger &= \hat{c}^\dagger\hat{d}^\dagger(\hat{c}\hat{d}\hat{c}^\dagger\hat{d}^\dagger)^{-1/2} = \hat{c}^\dagger\hat{d}^\dagger[(1+\hat{c}^\dagger\hat{c})(1+\hat{d}^\dagger\hat{d})]^{-1/2}, \end{aligned} \tag{26}$$

which, upon inspection, are just products of the SG phase operators of the two particles: $\hat{e}_k = \hat{e}_{\mathrm{SG}c}\hat{e}_{\mathrm{SG}d}$. Unlike in the HP case, the operators themselves do not depend on the Bargmann index. The corresponding phase states, assuming that the state is an eigenstate of $\hat{\Delta}$with eigenvalue $q \geqslant 0$, are then

$$|\eta, q\rangle = (1-|\eta|^2)^{1/2}\sum_{n=0}^{\infty}\eta^n|n+q\rangle_c \otimes |n\rangle_d, \tag{27}$$

where we have replaced k by q in our state labelling. In the case where $q = 0$ we obtain

$$|\eta, 0\rangle = (1-|\eta|^2)^{1/2}\sum_{n=0}^{\infty}\eta^n|n\rangle_c \otimes |n\rangle_d. \tag{28}$$

In the context of quantum optics, this is just the familiar two-mode squeezed vacuum state where $\eta = \mathrm{e}^{\mathrm{i}\phi}\tanh\theta$, θ being the squeeze parameter and ϕ the phase of the classical pump field. That the eigenstate of the two-mode phase operator, with $q = 0$, is the two-mode squeezed vacuum state has previously been noted by Agarwal [12].

Finally we consider the single-boson degenerate 'two-photon' realization of $su(1,1)$ where

$$\hat{K}_+ = \tfrac{1}{2}\hat{a}^{\dagger 2}, \qquad \hat{K}_- = \tfrac{1}{2}\hat{a}^2, \qquad \hat{K}_0 = \tfrac{1}{2}\left(\hat{a}^\dagger\hat{a} + \tfrac{1}{2}\right), \tag{29}$$

for which $\hat{C} = -(3/16)\hat{1}$. This implies that the possible Bargmann indices are $k = 1/4$ and $3/4$, the representations of the covering group $\widetilde{SU}(1,1)$. As can easily be checked, the representation of the former contains only the even boson number states while the latter only the odd. That is, $\mathcal{D}_{1/4}^+ : \left\{\left|\frac{1}{4}, m\right\rangle = |2m\rangle_a\right\}$ whereas $\mathcal{D}_{3/4}^+ : \left\{\left|\frac{1}{4}, m\right\rangle = |2m+1\rangle_a\right\}$. Thus, the Hilbert space of the boson states, $\mathcal{H} : \{|n\rangle_a; n = 0, 1, 2\ldots\}$, is the sum $\mathcal{H} = \mathcal{D}_{1/4}^+ \oplus \mathcal{D}_{3/4}^+$. The phase operators in these cases are

$$\hat{e}_{1/4} = \sum_{m=1}^{\infty}|2m-2\rangle_{aa}\langle 2m|, \qquad \hat{e}_{3/4} = \sum_{m=1}^{\infty}|2m-1\rangle_{aa}\langle 2m+1|, \tag{30}$$

and the corresponding phase states are

$$|\eta, 1/4\rangle = (1-|\eta|^2)^{1/2}\sum_{m=0}^{\infty}\eta^m|2m\rangle_a, \tag{31}$$

$$|\eta, 3/4\rangle = (1-|\eta|^2)^{1/2}\sum_{m=0}^{\infty}\eta^m|2m+1\rangle_a, \tag{32}$$

where we have used the bose number states $|n\rangle_a$ in the expansions. There is a connection between the SG phase operator and the two $SU(1,1)$ phase operators of equation (30). If we square the SG phase operator we obtain

$$\hat{e}_{\mathrm{SG}}^2 = \sum_{n=1}^{\infty}|n-1\rangle_{aa}\langle n+1| = \hat{e}_{1/4} + \hat{e}_{3/4}. \tag{33}$$

Thus, because of the splitting of $\mathcal{H}$ into even and odd subspaces, the phase states of equations (31) and (32) are also eigenstates of the square of the SG phase operator. Furthermore, these phase states can be shown to be even and odd superpositions of the phase states (8) of the SG phase operator:

$$|\eta, 1/4\rangle = \tfrac{1}{2}(1+|z|^2)^{1/2}(|z\rangle + |-z\rangle), \tag{34}$$

$$|\eta, 3/4\rangle = \tfrac{1}{2}(1+|z|^2)^{1/2}(|z\rangle - |-z\rangle), \tag{35}$$

where in both cases $\eta = z^2$. These are the phase state analogues of the usual even and odd coherent states, Schrödinger cat states, formed out of the ordinary coherent states $|\pm\alpha\rangle$ [14]. These 'phase cat' states have recently been discussed [15].

We wish to bring to the attention of the reader an extensive discussion of issues related to the Holstein–Primakoff realization of $SU(1,1)$ and of the associated phase states and phase operators for that group and related groups [16].

In summary, we have shown that the phase operators for the Lie algebra $su(1,1)$ rather generally provide a realization of the operator algebra associated with particles obeying infinite statistics. Various realizations of the algebra provide a variety of positive discrete representations for decomposing the $su(1,1)$ phase operators. We discussed three realizations, namely: (i) the Holstein–Primakoff realization of the $su(1,1)$ phase operators which for the case with $k = 1/2$ are just the Susskind–Glogower operators discussed in this context by Agarwal; (ii) the non-degenerate two-boson realization; and (iii) and the degenerate 'two-photon' realization. We discussed the phase coherent states relevant to each of these cases.

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References

[1] Agarwal G S 1991 *Phys. Rev.* A **44** 8398
[2] Greenberg O W 1990 *Phys. Rev. Lett.* **64** 705
[3] Susskind L and Glogower J 1964 *J. Phys.* **1** 49
[4] Aharonov Y, Lerner E C, Huang H W and Knight J M 1973 *J. Math. Phys.* **14** 746
[5] Shapiro J H and Shepard S R 1991 *Phys. Rev.* A **43** 3795
[6] Gerry C C 1988 *Phys. Rev.* A **38** 1734
[7] See, for example, Perlomov A 1986 *Generalized Coherent States and Their Applications* (Berlin: Springer)
[8] Vourdas A 1990 *Phys. Rev.* A **41** 1653
Vourdas A 1992 *Phys. Rev.* A **45** 1943
[9] Holstein T and Primakoff H 1940 *Phys. Rev.* **58** 1098
[10] Gerry C C 1983 *J. Phys. A: Math. Gen.* **16** L7
[11] Barut A O and Girardello L 1971 *Commun. Math. Phys.* **21** 41
[12] See, for example, Gerry C C 1991 *J. Opt. Soc.* B **8** 685 and references therein
[13] Agarwal G S 1993 *Opt. Commun.* **100** 479
[14] See, Bužek V and Knight P L 1995 *Progress in Optics* vol 34, ed E Wolf (Amsterdam: Elsevier) p 1
Gerry C C and Knight P L 1997 *Am. J. Phys.* **65** 964
[15] Gerry C C, Benmoussa A and Bruno K M 2003 *J. Opt. B: Quantum Semiclass. Opt.* **5** 109
[16] Kastrup H A 2003 *Fortsch. Phys.* **51** 975